# Classical and quantum codes, 2d CFTs and holography

Anatoly Dymarsky

University of Kentucky

ITMP seminar

March 6, 2024

# Bird's eye view

- code CFTs:: constructing (interesting) theories from codes

  Dolan, Goddard, Montague, hep-th/9410029

  AD with Shapere, 2009.01236, 2009.01244

- TQFTs describe (abelian) anyons, e.g. toric code

  state of 3d Chern-Simons = CFT conformal block

- holographic correspondence: 2d CFT= 3d bulk theory

  bulk gravity= ensemble of boundary theories

# Example: compact scalar CFT

compact scalar of radius $R$

- 2d CFT on a torus

$$S = \int d^2 z \, |\partial\phi|^2, \qquad \phi \sim \phi + R$$

- even self-dual lattice $\Lambda$ of moments and windings

$$\phi(z+1) = \phi(z) + nR, \qquad \phi(z+\tau) = \phi(z) + mR$$
$$p_L = \frac{n}{R} + \frac{mR}{2}, \qquad p_R = \frac{n}{R} - \frac{mR}{2}$$

- CFT is defined by $\Lambda \subset \mathbb{R}^{n,\bar{n}}$

# Classical additive codes and lattices

- exampole: binary additive code $\mathcal{C}$ = a collection of "codewords," vectors of length $n$ with elements in $\mathbb{Z}_2$

$$\mathsf{c} = (a_1, \ldots, a_n) \in \mathcal{C}, \qquad a_i \in \{0, 1\}$$

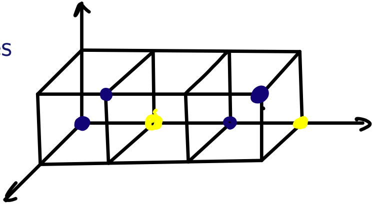
- Construction A: code-based lattices

$$\Lambda_{\mathcal{C}} = \{v/\sqrt{2} \,|\, v \bmod 2 \in \mathcal{C}\}$$

- codewords $\mathsf{c} \in \mathcal{C}$ – vertexes of the unit cube
- a good code: include as many codewords as possible, while keeping them as distinct as possible
- Hamming $[8, 4, 4]$ codes $\rightarrow$ root lattice $E_8$

# Codes and lattices

- binary codes → lattices



placing unit cubes at each even coordinate points

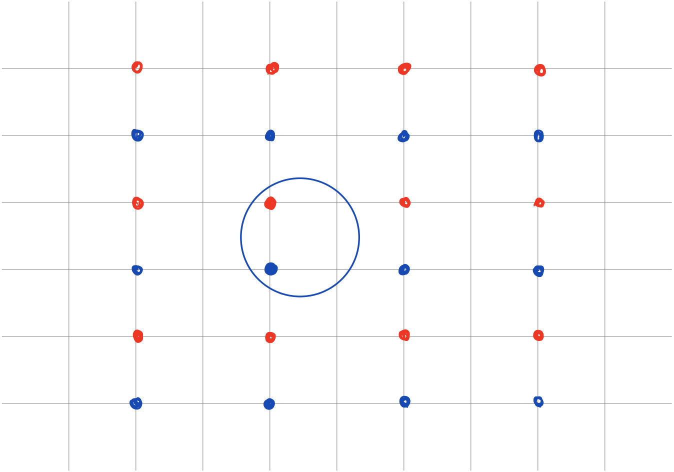
$$\Lambda(\mathcal{C}) = \{v/\sqrt{2}\,|\,v \bmod 2 \in \mathcal{C}\}$$

Construction A

- double-even code → even lattice
- self-dual code → self-dual lattice

# Codes and lattices

- Construction A



# Code controls the CFT spectrum

- CFT partition function $\propto$ lattice theta-function

$$Z_{\mathcal{C}} = \Theta_{\Lambda(\mathcal{C})}/\eta^n \bar{\eta}^{\bar{n}}$$

- code enumerator polynomial

$$\Theta_{\Lambda(\mathcal{C})} = W_{\mathcal{C}}(\Psi_c)$$

$\Psi_c$ – partial sums over sublattices (Jacobi theta)

- modular properties of $Z$ from $W$

enumerator $\rightarrow$ theta-function $\rightarrow$ partition function

modular invariance: MacWilliams identity of $W_{\mathcal{C}}$

# Code CFTs and modular bootstrap

codes with larger Hamming distance →

CFTs with larger spectral gap

- solutions to modular bootstrap, isospectral theories
  with Shapere 2009.01236, PRL 126 (16), 161602
- fake Z
  with Kalloor 2211.15699, JHEP 2023 (6), 1-29.
- linear programming bounds for codes, sphere packings, and CFTs

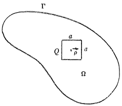
CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

"La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.



# Holography vs CS/RCFT

- holograohy: bulk=boundary

$$Z_{CFT} = \Psi_{\text{bulk}}, \quad \text{or} \quad \langle Z_{CFT} \rangle = \Psi_{\text{bulk}}$$

- CS/RCFT: Chern-Simons wavefunction = RCFT conformal block

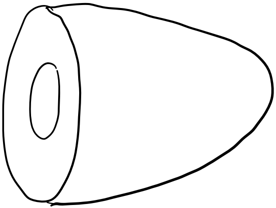
$$\chi_i = \Psi_i$$

- holography requires gauging (certain) symmetries – condensing (certain) anyons

$Z_{CFT} = \sum_{ij} M_{ij} \chi_i \bar{\chi}_j$

$M_{ij}$ is a sufrace operator

$\mathcal{M} = TAdS$



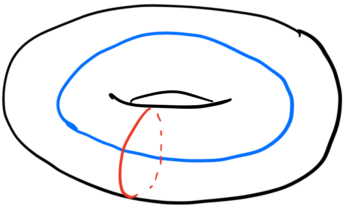
# Example: Kitaev's toric code

ground state of toric code = 2 qubits

- $e$, $m$ and $f$ anyons
- closed loops = Wilson line operators

$$\begin{aligned} \color{red}{e} &= Z \otimes I \quad \color{red}{m} = I \otimes Z \\ \color{blue}{e} &= I \otimes X \quad \color{blue}{m} = X \otimes I \end{aligned}$$

- $e$ or $m$ but not $f$ can be condensed



# Anyon condensation = stabilizer CSS code

- $e$ condenses = any loop of $e$ acts trivially

$$e = Z \otimes I \qquad e = I \otimes X$$

$$\Psi_e = (|00\rangle + |01\rangle)/\sqrt{2}$$

- $\Psi_e$ and $\Psi_m$ are (the only) modular invariants

$$Z_{R/\sqrt{2}}(\tau) = \Psi_e(\tau), \qquad Z_{\sqrt{2}R}(\tau) = \Psi_m(\tau)$$

- holographic duality: compact scalar CFT=abelian TQFT after anyon condensation

# The same in Chern-Simons language

- ground state physics of toric code = AB Chern-Simons

$$S = \frac{1}{2\pi} \int A \wedge dB$$

- arbitrariness of b.c.

$$A^{\pm} = \frac{R}{\sqrt{2}} A \pm \frac{\sqrt{2}}{R} B, \qquad \Psi(A^{+}_{\bar{z}}, A^{-}_{z})$$

- after gauging $e$ resulting 3d theory is topologically trivial

$$S = \frac{1}{4\pi} \int A \wedge dB$$

# General case

- $U(1)^n \times U(1)^{\bar{n}}$ Chern-Simons theory

$$S = \frac{1}{4\pi} \int_{\mathcal{M}} K(\vec{A}, d\vec{A})$$

- $K$ is the Gram matrix of even lattice $\Lambda_0 \subset \mathbb{R}^{n,\bar{n}}$
  Wilson lines are labeled by

$$\mathsf{c} \in \mathscr{D} = \Lambda_0^* / \Lambda_0$$

- Wilson lines wrapping $\Sigma = \partial\mathcal{M}$ act as "Pauli" matrices

$$W[\mathsf{c}_a, \mathsf{c}_b] \Psi_{\mathsf{c}'} = e^{2\pi i (\mathsf{c}_a, \mathsf{c}')} \Psi_{\mathsf{c}_b + \mathsf{c}'}$$

toric code: $\Lambda_0 = (\sqrt{2}\mathbb{Z})^2$, $\mathscr{D} = \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathsf{c} = (a, b)$, $|\mathsf{c}|^2 = 2ab$

# Even codes = condensable anyons

- anyone is condensable = Wilson loop $W_{\mathsf{c}}$ has trivial braiding

$$|c|^2/2 = 0 \bmod 2$$

toric code: $(0,1)$ and $(1,0)$ are even, but $(1,1)$ is not

- condensable anyons (non-anomalous subgroup) = even code
- maximal (Lagrangian) subgroup = even self-dual code
- condensation of $\mathcal{C}$ anyons $\rightarrow$ CS with even self-dual $\Lambda_{\mathcal{C}}$

$$\Lambda_{\mathcal{C}} = \{\mathsf{c} + v \,|\, \mathsf{c} \in \mathcal{C}, \;\; v \in \Lambda_0\}$$

- holography: code CFT = boundary theory of $\Lambda_{\mathcal{C}}$ CS

2310.06012, 2310.13044

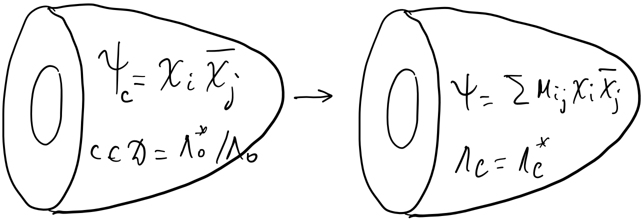
# Condensation process as a quantum code

- anyons $\mathsf{c} \in \mathcal{C}$ condense = Wilson loops $W_{\mathsf{c}}$ are trivial

  projector on trivial space

$$P_{\mathcal{C}} = \frac{1}{|\mathcal{C}|} \sum_{\mathsf{c}_a \in \mathcal{C}} \sum_{\mathsf{c}_b \in \mathcal{C}} W[\mathsf{c}_a, 0] W[0, \mathsf{c}_b] = |\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}|$$

  CSS symplectic code $\mathcal{C} \oplus \mathcal{C}$
- code CFT parition function $\Psi_{\mathcal{C}}$ = stablizer state of $\mathcal{C} \oplus \mathcal{C}$
- proector $P_{\mathcal{C}}$ is a surface operator in 3d TQFT



# Ensemble holography

- gravity in the bulk = ensemble of boundary CFTs

  averaging over Narain moduli space is holographically dual to "$U(1)$-gravity" – a sum over 3d topologies of (the pertubative part of) $U(1)^n \times U(1)^n$ Chern-Simons

  Maloney-Witten '2020

  Afkhami-Jeddi, Cohn, Hartman, Tajdini '2020

$$\langle Z_{CFT} \rangle \propto \sum_{\gamma \in SL(2,\mathbb{Z})} \frac{1}{|\eta(\gamma\,\tau)|^{2n}}$$

- bulk wormhole geometries

# Ensemble of code CFTs

- ensemble of code CFTs (ensemble of maximal gaugings of 3d TQFT) = "TQFT gravity" (sum over topologies)

$$\sum_{\mathcal{C}} Z_{\mathcal{C}} \equiv \sum_{\mathcal{C}} \Psi_{\mathcal{C}} \quad \propto \sum_{\gamma \in SL(2,\mathbb{Z})} \gamma(\Psi_0)$$

  2310.06012: codes over $\mathbb{Z}_p \times \mathbb{Z}_p$, reproduces "$U(1)$-gravity" in the limit of $p \to \infty$

  extention to arbitrary genus $\Sigma$, in progress

- extention to arbitrary ensemble of codes, covers all examples of "Narain CFT ensemble/CS gravity" in the literature

# What is topology?

wave-function of CS on a handlebody with shrinkable $a$-cycle

- example: toric code

$$\color{red}{e} = Z \otimes I \quad \color{red}{m} = I \otimes Z$$

- general case: $\Psi_a = \Psi_{\vec{0}} = |00\rangle$ is a stabilizer state
  Swingle 2016
- quantum CSS code $\mathscr{D} \oplus \mathscr{D}^*$

$$P_a = \frac{1}{|\mathscr{D}|} \sum_{\mathsf{c}_a \in \mathscr{D}} W[\mathsf{c}_a, 0] = |\Psi_0\rangle\langle\Psi_0|$$

# Ensemble holography for toric code

toric code Pauli group generators

$$\begin{aligned} e &= Z \otimes I \quad m = I \otimes Z \\ e &= I \otimes X \quad m = X \otimes I \end{aligned}$$

- code CFT states

$$\Psi_e = (|00\rangle + |01\rangle)/\sqrt{2}, \quad \Psi_m = (|00\rangle + |10\rangle)/\sqrt{2}$$

- topologies

$$\Psi_a = |00\rangle, \quad \Psi_b = (|00\rangle + |10\rangle + |01\rangle + |11\rangle)/2.$$
$$\Psi_{a+b} = (|00\rangle + |10\rangle + |01\rangle - |11\rangle)/2$$

- holograpy

$$\Psi_e + \Psi_m = (\Psi_a + \Psi_b + \Psi_{a+b})/\sqrt{2}$$

# Ensemble holography and Howe duality

- $\mathbb{Z}_p$ analog of toric code ($\mathbb{Z}_p$ AB gauge theory)

$$\Lambda_0 = \sqrt{p}\, I \subset \mathbb{R}^{n,n}, \quad \mathscr{D} = (\mathbb{Z}_p \times \mathbb{Z}_p)^n$$

- Hilbert space $\mathcal{H}$ on torus $= 2n$ qupits
- Hilbert space on genus $g$ Riemann surface $\Sigma$

$$\mathcal{H}^g = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$$

- "Clifford" group preserving "Pauli" group of Wilson Ls.

$$Sp(2g\, 2n, \mathbb{Z}_p) \supset O(n, n, \mathbb{Z}_p) \times Sp(2g, \mathbb{Z}_p)$$

$O(n, n, \mathbb{Z}_p)$ maps (classical) codes $\mathcal{C}$ into codes
$Sp(2g, \mathbb{Z}_p)$ modular group of $\Sigma$

$$\sum_{O(n,n)} \Psi^g_{O\mathcal{C}} \propto \sum_{\gamma \in \text{“}Sp(2g)''} \gamma(\Psi^g_0)$$

# Conclusions

- codes encode condesable anyons in 3d abelian CS

  give rise to code CFTs via anyon condensation

- mathematical relation between coding theory and CFTs
- holographic duality: ensemble of codes = "CS gravity"
- extension beyond Narain theories?

  connection between codes and non-invertible symmetries, non-abelian TQFTs

  ensemble holography for various anyon condensation: Virasoro TQFT, JT gravity, etc.